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Exponential nonlocal symmetries and nonnormal reduction of order

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Abstract

The conventional approach to double reduction of the order of an ordinary differential equation using Lie symmetries is via the normal subgroups of point symmetries. We show that, provided that one is prepared to use nonlocal symmetries, initial reduction by the nonnormal subgroup does not prevent the double reduction. We further illustrate our results with the general third-order equations invariant under the nonsolvable algebras, $sl(2, R)$ (of which the Chazy equation is a noted example) and $so(3)$.

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A common method for the investigation of the properties of ordinary differential equations is to examine them for Lie point symmetries (preferably using one of the better symbolic manipulation codes such as LIE devised by Head [11, 18] or the well-known interactive code of Nucci [15, 16]) and, when symmetries are found, to use the symmetries *inter alia* to reduce the order of the equation. If a sufficient number of symmetries 'of the right type' is available, the differential equation can be reduced to an algebraic equation.

In the conventional approach to the reduction of order of ordinary differential equations by means of their Lie symmetries the procedure laid down in the case of multiple symmetries is to reduce by means of the normal subgroup [17]. In terms of this approach not all of the elements of a nonsolvable algebra are available to be used for reduction. In the case of two point symmetries, Γ_1 and Γ_2, with

$$[\Gamma_1, \Gamma_2] = \Gamma_1, \tag{1}$$

Γ_1 being the normal subgroup. Reduction by Γ_1 leads to the descendant of Γ_2 being a point symmetry of the reduced equation. Were Γ_2 the first symmetry to be used to reduce the order of the differential equation, Γ_1 would be lost as a point symmetry of the reduced equation.

The first part of this paper is concerned with the reduction of order via the nonnormal subgroup of two-dimensional algebras as this is the order of algebra at which this question initially arises. We show that, as far as reduction of order is concerned, there is no need for the choice of normal subgroup. In the second part of this paper we look at two important nonsolvable algebras, $sl(2, R)$ and $so(3)$, and see that, again, reduction by the nonnormal subgroup presents no obstacle to reduction for those who are prepared to accept a wider class of symmetry than point or contact. It is with this wider class of symmetry that the results presented here are likely to have the most application.

In Lie's [14] classification of two-dimensional algebras there are four types of algebra, two Abelian and two solvable. The latter pair of algebras have the Lie bracket (1) and are distinguished by the proportionality or otherwise of their elements. The relevant properties of these four algebras are summarized in table 1. We emphasize that these four algebras are representatives of four distinct classes of algebras equivalent under point transformation. For reduction of order it does not matter which symmetry is used firstly if the algebra is Abelian and so we have no further interest in the Abelian algebras in this paper.

Table 1. Canonical forms of Lie algebras of dimension two and their properties.

Type	$[\Gamma_1, \Gamma_2]$	Nature	Canonical forms of Γ_1 and Γ_2
I	0	Abelian and unconnected	$\Gamma_1 = \partial_x$ $\Gamma_2 = \partial_y$
II	0	Abelian and connected	$\Gamma_1 = \partial_y$ $\Gamma_2 = x\partial_y$
III	Γ_1	Solvable and unconnected	$\Gamma_1 = \partial_y$ $\Gamma_2 = x\partial_x + y\partial_y$
IV	Γ_1	Solvable and connected	$\Gamma_1 = \partial_y$ $\Gamma_1 = y\partial_y$

We may use the Lie bracket relation (1) without loss of generality for, if we had

$$[X_1, X_2] = aX_1 + bX_2, \tag{2}$$

the definition

$$\Gamma_1 = aX_1 + bX_2 \qquad \Gamma_2 = aX_2 \tag{3}$$

(or an equivalent were a zero) would restore the Lie bracket relationship (1). We note further that in the case of one symmetry there is no loss of generality in taking it as ∂_x since all symmetries of the form

$$\Gamma = \xi(x, y)\partial_x + \eta(x, y)\partial_y \tag{4}$$

are equivalent to this under point transformation. (Equally a contact symmetry is equivalent to ∂_x under contact transformation.)

In table 1 the basic symmetry was taken by Lie to be ∂_y. Here we use ∂_x as we want the equations to be autonomous and so more in line with the appearance of the equations that we have treated in related articles [6, 8].

In the proportional case, type IV of table 1,

$$\Gamma_2 = f(x, y)\Gamma_1, \tag{5}$$

where $f(x, y)$ is some function. We take Γ_1 as ∂_x. Then (1) with (5) is $\partial f/\partial x = 1$, so that

$$f = x + g(y) \tag{6}$$

and we have

$$\Gamma_1 = \partial_x \qquad \Gamma_2 = [x + g(y)]\partial_x. \tag{7}$$

If we introduce the new variables $X = x + g(y)$ and $Y = y$, the symmetries become

$$\Gamma_1 = \partial_X \qquad \Gamma_2 = X\partial_X \tag{8}$$

which is a normal form for this class. By a normal form we mean a simple form for which all other members of the equivalence class are related by point transformation and which has the basic properties of its class. We revert to lower case variables.

To reduce order by Γ_2, we determine the two invariants of

$$\Gamma_2^{[1]} = x\partial_x + 0\partial_y - y'\partial_{y'} \tag{9}$$

from the solution of the associated Lagrange's system

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}y'}{-y'}. \tag{10}$$

We take the two invariants to be

$$u = y \qquad v = xy'. \tag{11}$$

The descendant of Γ_1 is

$$\bar{\Gamma}_1 = y'\partial_v = \frac{v}{x}\partial_v. \tag{12}$$

Since

$$\mathrm{d}u = y'\,\mathrm{d}x = \frac{v}{x}\,\mathrm{d}x, \tag{13}$$

we have

$$x = \exp\left[\int \frac{\mathrm{d}u}{v}\right] \tag{14}$$

and so (12) becomes

$$\bar{\Gamma}_1 = \exp\left[-\int \frac{\mathrm{d}u}{v}\right] v\partial_v \tag{15}$$

which is an exponential nonlocal symmetry. However, since

$$\bar{\Gamma}_1^{[1]} = \exp\left[-\int \frac{\mathrm{d}u}{v}\right] \left\{0\partial_u + v\partial_v + (v' - 1)\partial_{v'}\right\}, \tag{16}$$

the exponential terms in the associated Lagrange's system for the two invariants of $\bar{\Gamma}_1$ cancel to leave

$$\frac{\mathrm{d}u}{0} = \frac{\mathrm{d}v}{v} = \frac{\mathrm{d}v'}{v' - 1} \tag{17}$$

so that the two invariants are

$$p = u \qquad q = \frac{v' - 1}{v}. \tag{18}$$

Reduction of order by the transformation (18) is a well-defined operation. For example, the general second-order ordinary differential equation invariant under (8) (now written in lower case variables) is

$$y'' = y'^2 f(y). \tag{19}$$

Equation (19) is representative of the class of equations invariant under the representation of the type IV algebra used here. It is clearly trivially integrable (in the sense of reduction to quadratures) which is not surprising as it is linear in the representation given in table 1. However, our present interest is in reduction of order. We compare the reduction procedures for the two routes, namely, Γ_1 followed by Γ_2 and Γ_2 followed by Γ_1. We have

$$\begin{array}{llll} \Gamma_1: & u = y, v = y' & \Gamma_2: & u = y, v = xy' \\ & v' = vf(u) & & v' = vf(u) + 1 \\ \bar{\Gamma}_2: & p = u, q = v'v & \bar{\Gamma}_1: & p = u, q = (v'-1)/v \\ & q = f(p) & & q = f(p). \end{array} \tag{20}$$

In the nonproportional case we can again take Γ_1 as ∂_x and assume that

$$\Gamma_2 = \xi\partial_x + \eta\partial_y \tag{21}$$

in which we must have η nonzero. From (1)

$$\frac{\partial\xi}{\partial x} = 1 \qquad \frac{\partial\eta}{\partial x} = 0 \tag{22}$$

so that

$$\xi = x + f(y) \qquad \eta = g(y). \tag{23}$$

Under the change of variables

$$\begin{aligned} X &= x + f(y) - \exp\left[\int \frac{\mathrm{d}y}{g(y)}\right] \int f'(y) \exp\left[-\int_0^y \frac{\mathrm{d}u}{g(u)}\right] \mathrm{d}y \\ Y &= \exp\left[\int \frac{\mathrm{d}y}{g(y)}\right] \end{aligned} \tag{24}$$

the two symmetries take the normal forms

$$\Gamma_1 = \partial_X \qquad \Gamma_2 = X\partial_X + Y\partial_Y. \tag{25}$$

We revert to lower case variables and use the normal form (25).

For reduction of order via the nonnormal subgroup Γ_2, the variables are found from the solution of

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}y'}{0} \tag{26}$$

to be $u = y/x$ and $v = y'$ so that

$$\bar{\Gamma}_1 = -\frac{y}{x^2}\partial_u = -\frac{1}{x}u\partial_u = -\exp\left[-\int \frac{\mathrm{d}u}{v-u}\right] u\partial_u \tag{27}$$

since

$$\mathrm{d}u = \left(\frac{y'}{x} - \frac{y}{x^2}\right)\mathrm{d}x = (v-u)\frac{\mathrm{d}x}{x}. \tag{28}$$

Reduction of order via $\bar{\Gamma}_1$ is achieved by means of the change of variables obtained from the solution of

$$\frac{\mathrm{d}u}{u} = \frac{\mathrm{d}v}{0} = \frac{(v-u)\,\mathrm{d}v'}{-v'(v-2u)}, \tag{29}$$

namely,

$$p = v \qquad q = uv'(v-u). \tag{30}$$

The general second-order ordinary differential equation invariant under (25) is

$$yy'' = f(y'). \tag{31}$$

The two routes for reduction, as indicated above, are

$$\begin{array}{llll} \Gamma_1: & u = y, v = y' & \Gamma_2: & u = y/x, v = y' \\ & uvv' = f(v) & & v' = f(v)/[u(u-v)] \\ \bar{\Gamma}_2: & p = v, q = uv' & \bar{\Gamma}_1: & p = v, q = uv'(v-u) \\ & q = f(p)/p & & q = f(p). \end{array} \tag{32}$$

We conclude that in the case of two-dimensional algebras reduction of order via the normal subgroup is not essential for the further reduction of order by the second element of the algebras. The reason for this is that, when the nonnormal subgroup is used, the second symmetry becomes exponential nonlocal and such a symmetry is as good as a point symmetry for reduction of order. This is another instance [9, 3, 10] of the utility of nonlocal symmetries in the reduction of order for ordinary differential equations.

The Chazy equation [5]

$$y''' + yy'' - \frac{3}{2}y'^2 = 0, \tag{33}$$

an instance of the generalized Chazy equation [6]

$$y''' + yy'' + ky'^2 = 0, \tag{34}$$

has the three Lie point symmetries

$$\Gamma_1 = \partial_x \qquad \Gamma_2 = x\partial_x - y\partial_y \qquad \Gamma_3 = x^2\partial_x + (12 - 2xy)\partial_y \tag{35}$$

which constitute a representation of the nonsolvable algebra $sl(2,R)$. Equation (33) is a member of the class of equations

$$\begin{aligned} y''' + yy'' = &-\frac{y^2}{96}(24y' + y^2) + (12y' + y^2)^2 \\ &\times F\left[\frac{y''}{(12y' + y^2)^{3/2}} + \frac{1}{72}\left(\frac{3y}{(12y' + y^2)^{1/2}} - \frac{y^3}{(12y' + y^2)^{3/2}}\right)\right], \end{aligned} \tag{36}$$

where F is an arbitrary function of its argument, invariant under the representation of $sl(2, R)$ in (35). We note that the numerical value $F = 1/96$ corresponds to the Chazy equation. The Chazy equation, be it the specific equation (33) or the generalized equation (34), is not strictly equivalent to (36) since the complete symmetry group [12] of either equation must contain four elements [4] whereas (36) is completely specified by the three elements of $sl(2, R)$ given in (35). The additional symmetry removes the arbitrariness of the function F.

The form of (36) is found by successive imposition of the requirements of invariance of the three symmetries in (35). If we commence with the arbitrary form

$$y''' = f(x, y, y', y''), \tag{37}$$

invariance under Γ_1 implies that f in (37) is free of x. The third extension of Γ_2 is

$$\Gamma_3^{[3]} = x\partial_x - y\partial_y - 2y'\partial_{y'} - 3y''\partial_{y''} - 4y'''\partial_{y'''} \tag{38}$$

and, when it is applied to (37) without x being present in f (due to the action of Γ_1), (37) is required to take the form

$$y''' = y^4\bar{f}\left(\frac{y'}{y^2}, \frac{y''}{y^3}\right). \tag{39}$$

The form (36) follows from the requirement that (39) is invariant under the action of the third extension of Γ_3. Since

$$[\Gamma_1, \Gamma_2] = \Gamma_1 \qquad [\Gamma_1, \Gamma_3] = 2\Gamma_2 \qquad [\Gamma_2, \Gamma_3] = \Gamma_3, \tag{40}$$

the conventional approach would have us reduce the order of the equation by either of Γ_1 or Γ_3, certainly not Γ_2. For Γ_1 the invariants are $u = y$ and $v = y'$ and Γ_2 and Γ_3 become, respectively,

$$\begin{aligned}\bar{\Gamma}_2 &= -u\partial_u + 2v\partial_v \\ \bar{\Gamma}_3 &= (12 - 2xy)\partial_u - (2y + 4xy')\partial_v \\ &= \left(12 - 2u\int\frac{\mathrm{d}u}{v}\right)\partial_u - \left(2u + 4v\int\frac{\mathrm{d}u}{v}\right)\partial_v\end{aligned} \tag{41}$$

since $\mathrm{d}u = y'\,\mathrm{d}x = v\,\mathrm{d}x$.

We observe that $\bar{\Gamma}_3$ is nonlocal and cannot be used for reduction of order since the nonlocal element is not exponential. The invariants of $\bar{\Gamma}_2$ are $p = v/u^2$ and $q = v'/u$. Under this change of variables the nonlocal symmetry $\bar{\Gamma}_3$ becomes

$$\bar{\bar{\Gamma}}_3 = -2\exp\left[-\int\frac{\mathrm{d}p}{q-2p}\right]\left[(12p+1)\partial_p + \left(6q + 3\frac{q}{p}\right)\partial_q\right] \tag{42}$$

which *is* exponential nonlocal and so, available for a further reduction of order.

We now consider the nonnormal route commencing with Γ_2. The invariants of Γ_2 are $u = xy$ and $v = x^2y'$ and Γ_1 and Γ_3 become, respectively,

$$\begin{aligned}\bar{\Gamma}_1 &= y\partial_u + 2xy'\partial_v \\ &= \exp\left[-\int\frac{\mathrm{d}u}{u+v}\right]\{u\partial_u + 2v\partial_v\} \\ \bar{\Gamma}_3 &= \exp\left[\int\frac{\mathrm{d}u}{u+v}\right]\{(12-u)\partial_u - 2(u+v)\partial_v\}\end{aligned} \tag{43}$$

since $\mathrm{d}u = (y + xy')\,\mathrm{d}x = [(u+v)/u]\,\mathrm{d}x$. As both $\bar{\Gamma}_1$ and $\bar{\Gamma}_3$ are exponential nonlocal, both are available for reduction of order. If we take $\bar{\Gamma}_1$, its invariants are found from the solution of

$$\frac{\mathrm{d}u}{u} = \frac{\mathrm{d}v}{2v} = \frac{(u+v)\,\mathrm{d}v'}{-2v + v'(2u+v)} \tag{44}$$

and are

$$p = \frac{v}{u^2} \qquad q = \frac{(u+v')v' - 2v}{u^3}. \tag{45}$$

After some calculation we find that

$$\begin{aligned}\bar{\bar{\Gamma}}_3 &= -\frac{2}{u}\exp\left[\int\frac{\mathrm{d}u}{u+v}\right]\left[(12p+1)\partial_p + 3(6q+p)\partial_q\right] \\ &= -2\exp\left[\int\frac{p\,\mathrm{d}p}{q-2p^2}\right]\left[(12p+1)\partial_p + 3(6q+p)\partial_q\right]\end{aligned} \tag{46}$$

since

$$\frac{\mathrm{d}p}{q-2p^2} = \frac{\mathrm{d}u}{1+up} \tag{47}$$

and

$$\frac{1}{u}\exp\left[\int\frac{\mathrm{d}u}{u+v}\right] = \exp\left[-\int\frac{\mathrm{d}u}{1+up}\right]. \tag{48}$$

The symmetry $\bar{\bar{\Gamma}}_3$ is exponential nonlocal and can be used for reduction of order. Its invariants are

$$\begin{aligned} r &= \frac{1}{72\zeta^3}\,[72q + 3p + 2] \\ s &= \frac{1}{\zeta^2}\left[q'\left(\zeta^2 - 72\zeta r + 1\right) - 18r\zeta^3 - 54r\zeta + 1\right], \end{aligned} \tag{49}$$

where $12p + 1 = \zeta^2$.

Under the successive reductions of order, the general equation (36) reduces to

$$s + 72F(r) = 0 \tag{50}$$

and the Chazy equation (33) to the simple algebraic equation

$$4s + 3 = 0. \tag{51}$$

It is of interest to note that the exponential nonlocal symmetry $\bar{\Gamma}_3$ remains exponential nonlocal under the reduction of order produced by transformation (45). We remark that there appears to be no foundation for the proposed requirement that reduction of order must be by the normal subgroup. The admission of a wider variety of symmetries for the reduction of order not only increases the chances for complete reduction of an equation but also improves the probability that type II hidden symmetries [1, 2] will arise and thereby further enhance the likelihood of complete reduction of order.

The classic nonsolvable algebra of dimension three is *so* (3). It occurs in physics whenever there is conservation of angular momentum. In that context it has a representation in terms of the polar and azimuthal angles of spherical coordinates. There are also representations in terms of vector fields in the plane. One of these is [7]

$$\begin{aligned} X_1 &= w\partial_t - t\partial_w \\ X_2 &= \left(1 + t^2 - w^2\right)\partial_t + 2tw\partial_w \\ X_3 &= 2tw\partial_t + \left(1 - t^2 + w^2\right)\partial_w. \end{aligned} \tag{52}$$

However, for the purpose of reduction it is necessary to take linear combinations of these symmetries so that the reduced symmetries will be exponential nonlocal. Under the combinations and transformations

$$\begin{aligned} &\Gamma_1 \leftarrow X_1 && x \leftarrow t + \mathrm{i}w \\ &\Gamma_2 \leftarrow X_2 + \mathrm{i}X_3 && y \leftarrow t - \mathrm{i}w \\ &\Gamma_3 \leftarrow X_2 - \mathrm{i}X_3 && \end{aligned}$$

we obtain the representation suitable for our purpose, namely,

$$\Gamma_1 = x\partial_x - y\partial_y \qquad \Gamma_2 = x^2\partial_x + \partial_y \qquad \Gamma_3 = \partial_x + y^2\partial_y. \tag{53}$$

The invariants of Γ_1 are $u = xy$ and $v = x^2y'$. In terms of these new variables Γ_2 and Γ_3, respectively, take the exponential nonlocal forms

$$\begin{aligned} \bar{\Gamma}_2 &= \exp\left[\int \frac{\mathrm{d}u}{u + v}\right](1 + u)\partial_u \\ \bar{\Gamma}_3 &= \exp\left[-\int \frac{\mathrm{d}u}{u + v}\right](1 + u)\,[u\partial_u + 2u\partial_v]. \end{aligned} \tag{54}$$

For reduction by $\bar{\Gamma}_2$, the new variables are

$$p = v \qquad q = v'(u + 1)(u + v) \tag{55}$$

and $\bar{\Gamma}_3$ maintains its exponential nonlocal form as

$$\bar{\bar{\Gamma}}_3 = \exp\left[\int \frac{(p-1)\mathrm{d}p}{q}\right]\left\{2p\partial_p + [2p(p-1)+3q]\partial_q\right\}. \tag{56}$$

Finally the new variables from $\bar{\bar{\Gamma}}_3$ are

$$\begin{aligned} r &= p^{-3/2}[q-2p(p+1)] \\ s &= 12 + \frac{4}{p} + 8p - p^2 + \frac{5r}{\sqrt{p}} + 3r\sqrt{p} + \frac{3r^2}{2} - 6\log p. \end{aligned} \tag{57}$$

The form of the reduced general equation is the algebraic equation

$$s = F(r), \tag{58}$$

where F is an arbitrary function of its arguments. In terms of the original coordinates, the most general third-order equation invariant under $so(3)$ is

$$y''' = \frac{6y''(xy'-y)}{xy+1} - \frac{6y'(xy'-y)^2 + 12y'^2}{(xy+1)^2} + \frac{y'^2}{(xy+1)^2}F\left(\frac{xy+1-2x^2-2x^4y'^2}{x^4y'^2}\right). \tag{59}$$

Once again we see that reduction of order need not proceed by the normal subgroup since the nonlocal symmetries which arise are always exponential nonlocal symmetries. The acceptance of the useful rôle which nonlocal symmetries can play in reduction of order broadens the class of equations which can be reduced to algebraic form.

In conclusion we note that the initiation of the process of reduction of order need not be in a point (contact) symmetry. Consider the instance of the equation

$$y''' + yy'' + y'^2 = 0 \tag{60}$$

which is a particular form of the generalized Chazy equation (34). In addition to the obvious Lie point symmetries of invariance under translation in x and self-similarity (the first two symmetries of (35)), nonlocal symmetries of the form $\eta\partial_y$ are easily calculated from the determining equation [13]

$$\eta''' + \eta y'' + \eta'' y + 2\eta' y' = 0 \tag{61}$$

to be

$$\begin{aligned} \Gamma_4 &= \exp\left[-\int y\,\mathrm{d}x\right]\partial_y \\ \Gamma_5 &= \exp\left[-\int y\,\mathrm{d}x\right]\int \exp\left[\int y\,\mathrm{d}x\right]\partial_y \\ \Gamma_6 &= \exp\left[-\int y\,\mathrm{d}x\right]\int x\exp\left[\int y\,\mathrm{d}x\right]\partial_y. \end{aligned} \tag{62}$$

The invariants obtained from the associated Lagrange's system for Γ_4 are $u = x$ and $v = y' + y^2/2$. The reduced equation of the second order is

$$\frac{\mathrm{d}^2v}{\mathrm{d}u^2} = 0 \tag{63}$$

which is trivially solved. The solution of (60) follows from the solution of the resulting Riccati equation

$$y' + \frac{1}{2}y^2 = A + Bx, \tag{64}$$

where A and B are arbitrary constants of integration. This is an instance in which not only the reduction of order can be easily obtained by means of the use of a nonlocal symmetry but also the solution of the original differential equation can equally easily be obtained.

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